THE ALGEBRAIC NUMBERS DEFINABLE IN VARIOUS EXPONENTIAL FIELDS

JONATHAN KIRBY, ANGUS MACINTYRE, AND ALF ONSHUUS

1. Introduction

An exponential field (or E-field) is a field, F, of characteristic 0, together with $\mathrm{E} : F \to F$ satisfying

- $\mathrm{E}(0) = 1$
- $\mathrm{E}(x + y) = \mathrm{E}(x) \cdot \mathrm{E}(y)$.

Every mathematician knows the classical E-fields $\mathbb{R}$ and $\mathbb{C}$. There are also the LE-series (see [14]), and the surreal numbers [1].

More recently, Zilber has produced beautiful "complex" examples [16]. In $\mathbb{C}$, the kernel of the exponential map is $2\pi i\mathbb{Z}$, an infinite cyclic group. In addition, $\mathbb{C}$ is algebraically closed, and its exponential map is surjective. Zilber considered E-fields with these properties, which also satisfy the conclusion of Schanuel's conjecture (see 3.2 below), and which are *strongly exponentially-algebraically closed*, an analogue of being algebraically closed, but taking into account the exponentiation (see 3.4 below). In this paper we call such E-fields *Zilber fields*. (Other papers use this name for slightly larger or smaller classes of exponential fields, but the distinction is not important for our purposes.) There is an excellent exposition of these E-fields by Marker [11], and a detailed exposition in [3].

The complex exponential field $\mathbb{C}$ also has the property that for any countable subset $X \subseteq \mathbb{C}$, there are only countably many $a \in \mathbb{C}$ which are exponentially algebraic over X. This is the *countable closure property (CCP)* (see 3.3 below, or [4] for more details of exponential algebraicity). Zilber proved the dramatic result that there is a unique Zilber field (we call it $\mathbb{B}$) of cardinality $2^{\aleph_0}$, which satisfies the countable closure property. He has made the profoundly explanatory conjecture that $\mathbb{B} \cong \mathbb{C}$.

Much is known about the logic of these examples. The real E-field $\mathbb{R}$, the LE-series field, and the surreal numbers are elementarily equivalent E-fields ([14], [13], and [8]). They are model-complete, and decidable if Schanuel's conjecture is true ([15], [10]).

It follows from Gödel's incompleteness theorem that $\mathbb{C}$ is undecidable (see e.g. [12]), and it is not model-complete ([9], [11]). The same undecidability argument works for Zilber's E-fields, and a different argument shows the failure of model-completeness [3].

In this paper we consider, for each example above, the issue of which algebraic numbers are pointwise definable. For the real cases the problem is trivial, since one already knows that in their pure field theory one can define all real algebraic numbers [12]. The same question (understanding the pointwise definable points) for the complex exponential field had already been asked by Mycielski.

In the "complex" cases the notion of real abelian algebraic number is central (see 2.1 below). The main theorems are:
Theorem 1: For any E-field with cyclic kernel, in particular $\mathbb{C}$ or the Zilber fields, all real abelian algebraic numbers are pointwise definable.
Theorem 2: For the Zilber fields, the only pointwise definable algebraic numbers are the real abelian numbers.

The conjecture of Zilber above is one of two main open questions around the complex exponential field, the other being whether the real subfield is (setwise) definable. They cannot both have a positive answer, as can be seen for example from Theorem 2. One step towards Zilber's conjecture would be to show that Theorem 2 holds for the complex exponential field. One might hope this would be easier than the full conjecture, but we have not been able to prove it even assuming Schanuel's Conjecture.

2. Defining the real abelian numbers

2.1. $\mathbb{Q}^{\mathrm{rab}}$. In this section we consider E-fields F where $\mathrm{Ker} := \{x \in F : \mathrm{E}(x) = 1\}$ is an infinite cyclic group. Let τ and $-\tau$ be the generators.

Note that $\left\{\mathrm{E}\left(\frac{j\tau}{n}\right) : j = 0, \ldots, n-1\right\}$ are distinct nth roots of 1, so $F \supset U$, the group of *all* roots of unity. Thus

$$F \supset \mathbb{Q}^{ab} = \mathbb{Q}(U) = \mathbb{Q}[U],$$

the maximal abelian extension of $\mathbb{Q}$. Let $\mathbb{Q}^{\mathrm{alg}}$ be the field-theoretic algebraic closure of $\mathbb{Q}$ (as an abstract field).

It is important to note that in no algebraically closed field F of characteristic 0 is there a unique subfield $L \subsetneq F$ with $F = L(i)$. It follows by Artin-Schreier theory (see [2]) that there always is at least one such L. For if F has transcendence degree κ over $\mathbb{Q}$, pick a transcendence basis B over $\mathbb{Q}$ of cardinality κ and let L be a maximal formally real extension of $\mathbb{Q}(B)$ in F. L will be real-closed. Indeed, by Artin-Schreier, if F is a finite proper extension of any subfield L', then $F = L'(i)$ and L' is real closed. Note too that $L = \mathrm{Fix}(\sigma)$, where σ is an involution of $\mathrm{Aut}(F)$. Conversely, the fixed field of any involution of F is a field L with $F = L(i)$.

An elaboration of such arguments naturally gives an isomorphism between conjugacy classes of involutions of $\mathrm{Aut}(F)$ and isomorphism types of real-closed fields of transcendence degree κ over $\mathbb{Q}$.

Let us apply these ideas to $L = \mathbb{Q}^{\mathrm{alg}}$. Any K with $L = K(i)$ is isomorphic to the field of real algebraic numbers, so there is just one conjugacy class of involutions in $\mathrm{Aut}(\mathbb{Q}^{\mathrm{alg}})$. There are, however, $2^{\aleph_0}$ many involutions in this conjugacy class. This is because $\mathbb{Q}\left(\left\{\sqrt{p} \mid p \text{ prime}\right\}\right)$ has $2^{\aleph_0}$ different orderings (you can choose, independently for each p, which $\sqrt{p}$ is positive), and the corresponding real closures are distinct (but isomorphic). For example, pick a $\sqrt{p}$. In some real closures this will be a square, in others $-\sqrt{p}$ will be a square.

Finally, note that all restrictions to $\mathbb{Q}^{\mathrm{ab}}$ of involutions in $\mathrm{Aut}(\mathbb{Q}^{\mathrm{alg}})$ will be the same involution σ_0, characterized by $\sigma_0(x) = x^{-1}$ for all $x \in U$. We call the elements of $\mathrm{Fix}(\sigma_0)$ the *real abelian numbers*, and write $\mathbb{Q}^{\mathrm{rab}}$ for $\mathrm{Fix}(\sigma_0)$. We will prove in 2.7 that every element in $\mathbb{Q}^{\mathrm{rab}}$ is a rational combination of special values of the cosine function, which are totally real, so $\mathbb{Q}^{\mathrm{rab}}$ is totally real. This implies that it is included in any maximal formally real subfield of $\mathbb{Q}^{\mathrm{ab}}$. Now $\mathbb{Q}^{\mathrm{rab}}$ has only the one extension in $\mathbb{Q}^{\mathrm{ab}}$, and that is not formally real, so $\mathbb{Q}^{\mathrm{rab}}$ can alternatively

be characterized as the unique maximal formally real subfield of $\mathbb{Q}^{\mathrm{ab}}$, or as the intersection of $\mathbb{Q}^{\mathrm{ab}}$ with the field $\mathbb{Q}^{\mathrm{tr}}$ of totally real numbers.

2.2. **Defining $\mathbb{Z}$.** We can define $\mathbb{Z}$ as

$$\{y : \forall x[E(x) = 1 \to E(yx) = 1]\},$$

a $\forall$-definition.

We can define $\mathbb{Q}$ as

$$\{y : (\exists z, w \in \mathrm{Ker})[z = wy]\},$$

an $\exists$-definition.

In $\mathbb{C}$, there is also an $\exists$-definition of $\mathbb{Z}$ given by Laczkovich [7]. He used the idea

$$x \in \mathbb{Z} \iff (x \in \mathbb{Q} \wedge 2^x \in \mathbb{Q})$$

but one has to pay attention to the ambiguity in 2^x, and, in the general case, to the *existence* of logarithms. Consider the formula $\Theta(x)$ defined by

$$\exists t \left[\mathrm{E}(t) = 2 \wedge \mathrm{E}(xt) \in \mathbb{Q} \wedge x \in \mathbb{Q}\right].$$

Lemma. *Suppose $F \models (\exists t)[E(t) = 2]$. Then $F \models \Theta(x)$ if and only if $x \in \mathbb{Z}$.*

Proof. Suppose $F \models \Theta(x)$. Then $x = \frac{m}{n}$ with $m, n \in \mathbb{Z}, n > 0$. Then $\mathrm{E}(xt) = \mathrm{E}\left(\frac{mt}{n}\right)$ and $\mathrm{E}(xt)^n = 2^m$. But $\mathrm{E}(xt) \in \mathbb{Q}$, so $\frac{m}{n} \in \mathbb{Z}$, that is, $x \in \mathbb{Z}$.

Conversely, suppose $x \in \mathbb{Z}$, and $\mathrm{E}(t) = 2$. Then $\mathrm{E}(tx) = 2^x \in \mathbb{Q}$. $\square$

Thus if 2 has a logarithm in F, $\mathbb{Z}$ has a $\exists$-definition. A similar argument works if any prime number has a logarithm.

2.3. **Defining $\{\tau, -\tau\}$.** This two element set is defined by

$$x \in \{\tau, -\tau\} \Leftrightarrow ((x \in \mathrm{Ker}) \wedge ((\forall y \in \mathrm{Ker})\, (\exists n \in \mathbb{Z})\, [nx = y])).$$

The complexity of this definition is $\forall\exists\forall$ for a general F, but only $\forall\exists$ if some prime has a logarithm.

2.4. **Sine and cosine.** We are not able to distinguish i from $-i$ in the complex exponential case. But we can define cosine and sine there, and the same definitions make sense in any exponential field in which -1 is a square, namely:

$$\begin{aligned} \cos(x) = y \quad &\Leftrightarrow \quad (\exists j) \left[j^2 = -1 \wedge y = 1/2\left(\mathrm{E}\left(jx\right) + \mathrm{E}\left(-jx\right)\right)\right] \\ &\Leftrightarrow \quad (\forall j) \left[j^2 = -1 \to y = 1/2\left(\mathrm{E}\left(jx\right) + \mathrm{E}\left(-jx\right)\right)\right] \end{aligned}$$

and

$$\begin{aligned} \sin(x) = y \quad &\Leftrightarrow \quad (\exists j) \left[j^2 = -1 \wedge y = 1/2j\left(\mathrm{E}\left(jx\right) - \mathrm{E}\left(-jx\right)\right)\right] \\ &\Leftrightarrow \quad (\forall j) \left[j^2 = -1 \to y = 1/2j\left(\mathrm{E}\left(jx\right) - \mathrm{E}\left(-jx\right)\right)\right]. \end{aligned}$$

Thus the graphs of cosine and sine are both $\exists$- and $\forall$-definable. The standard results of elementary trigonometry are easily proved (just using that E is a homomorphism), for example:

i. $\cos(-x) = \cos(x)$;
ii. $\sin(-x) = -\sin(x)$;
iii. if $j^2 = -1$, $\{x : \sin(x) = 0\} = \frac{1}{2j}\mathrm{Ker}$;
iv. if $j^2 = -1$, $\{x : \cos(x) = 0\} = \left(\frac{1}{4j}\mathrm{Ker} \setminus \left(\frac{1}{2j}\mathrm{Ker}\right)\right)$;
v. if $j^2 = -1$, exactly one of $\sin(\alpha/4j)$ and $\sin(-\alpha/4j)$ is 1 and the other is -1 for any $\alpha \in \mathrm{Ker} \setminus 2\mathrm{Ker}$.

2.5. **Defining π.** We give a definition which is correct for the complex exponential, and has an unambiguous meaning for any exponential field with cyclic kernel.

From the definition of $\{\tau, -\tau\}$ we get an unambiguous definition of $\{\frac{\tau}{2j}, \frac{-\tau}{2j}\}$ for any j such that $j^2 = -1$. Think of this two element set as $\{\pi, -\pi\}$ and define π as the unique element t of this set with $\sin(t/2) = 1$. The other element is then $-\pi$.

2.6. **Separating $\pm\sqrt{2}$ (for example).** $\sqrt{2} = 2\frac{1}{\sqrt{2}}$, and $\cos\left(\frac{\pi}{4}\right) = +\frac{1}{\sqrt{2}}$, at least in $\mathbb{C}$. We *define* in general $+\sqrt{2} = 2\cos\left(\frac{\pi}{4}\right)$.

2.7. **Pointwise definition of elements of $\mathbb{Q}^{\mathrm{rab}}$.** Let $\alpha \in \mathbb{Q}^{\mathrm{ab}}$. Then $\alpha \in \mathbb{Q}[U]$, so it can be expressed as a finite sum:

$$\alpha = \sum r_n \mathrm{E}\left(s_n \tau\right),$$

with $r_n \in \mathbb{Z},\ s_n \in \mathbb{Q}$.

Recall that σ_0 is the involution in $\mathrm{Aut}(\mathbb{Q}^{\mathrm{ab}})$ characterized by $\sigma_0(x) = x^{-1}$ for all $x \in U$. Then if $\alpha \in \mathbb{Q}^{\mathrm{rab}}$ we have

$$\alpha = \left(\alpha + \sigma_0\left(\alpha\right)\right)/2 = \sum r_n \left(\frac{\mathrm{E}\left(s_n\tau\right) + \mathrm{E}\left(-s_n\tau\right)}{2}\right) = \sum r_n \cos(2\pi s_n)$$

which is clearly pointwise definable. This proves Theorem 1.

3. The other direction: Zilber fields.

3.1. **Partial exponential fields.** It is convenient to consider subfields of an exponential field which are not closed under exponentiation. Thus we define a *partial exponential field* to be a field F (of characteristic zero) together with a $\mathbb{Q}$-linear subspace $D(F)$ of F and a map $\mathrm{E} : D(F) \to F$ which satisfies

- $\mathrm{E}(0) = 1$
- $\mathrm{E}(x + y) = \mathrm{E}(x) \cdot \mathrm{E}(y)$.

If F is a partial exponential field then we say it is *generated* by a subset X if and only if $X \cap D(F)$ spans $D(F)$ and F is generated as a field by $X \cup \mathrm{E}(D(F))$. In particular, we have the notion of F being *finitely-generated* if a finite such X exists.

An *embedding* of partial exponential fields $\varphi : F \to K$ is a field embedding such that, given any $\alpha, \beta \in F$, if $E_F(\alpha) = \beta$ then $E_K(\varphi(\alpha)) = \varphi(\beta)$. We will say that F is a partial exponential subfield of K if it is a subset and the inclusion map is an embedding of partial exponential fields. Notice that $\mathbb{Q}$ with $D(\mathbb{Q}) = \{0\}$ is a partial exponential subfield of every partial exponential field. We call it $\mathbb{Q}_0$.

For another example, consider the subfield $SK = \mathbb{Q}^{\mathrm{ab}}(2\pi i)$ of $\mathbb{C}$, with $D(SK) = \mathbb{Q}\cdot 2\pi i$, and the restriction of the complex exponential map. (SK stands for *standard kernel*.) Then SK is generated as a partial exponential field by the single element $2\pi i$ because $\mathrm{E}(D(SK)) = U$, the roots of unity. Clearly SK is not finitely-generated as a pure field, because it contains $\mathbb{Q}^{\mathrm{ab}}$.

3.2. **Strong embeddings and Schanuel's Conjecture.** Suppose F is any exponential field, F_0 is a partial exponential subfield of F and let $Y \subset F$.

We will denote by $\mathrm{trans.deg.}(Y/F_0)$ the (algebraic) transcendence degree of the field extension $F_0(Y)/F_0$ and by $\mathrm{lin.dim.}_{\mathbb{Q}}(X/Y)$ the (linear) dimension of the $\mathbb{Q}$-vector space spanned by $X \cup Y$, quotiented by the subspace spanned by Y.

We say that F_0 is *strongly embedded* in F, and write $F_0 \lhd F$ if and only if for every finite subset $X \subseteq F$ we have

$$\text{trans.deg.}(X, \mathrm{E}(X)/F_0) \geqslant \text{lin.dim.}_{\mathbb{Q}}(X/D(F_0)).$$

For example, $\mathbb{R}$ is not strongly embedded in $\mathbb{C}$, because, taking $X = \{i\}$, we have trans.deg.$(i, e^i/\mathbb{R}) = 0$ and lin.dim.$_{\mathbb{Q}}(i/\mathbb{R}) = 1$.

We will say that a partial E-field F satisfies the *Schanuel Condition* (*SC*) if, whenever $\alpha_1, \ldots, \alpha_n$ in F are $\mathbb{Q}$-linearly independent, the transcendence degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, E(\alpha_1), \ldots, E(\alpha_n))$ over $\mathbb{Q}$ is greater than or equal to n. This is equivalent to saying that, for any finite $X \subseteq F$,

$$\text{trans.deg.}(X, \mathrm{E}(X)/\mathbb{Q}) \geqslant \text{lin.dim.}_{\mathbb{Q}}(X/0),$$

so it can be equivalently stated as $\mathbb{Q}_0 \lhd F$ (where, as mentioned before, $\mathbb{Q}_0$ is the partial E-field $\mathbb{Q}$ with trivial exponential domain).

The Schanuel condition implies that any nonzero kernel element is transcendental over $\mathbb{Q}$, something which is not always true in exponential fields (see Section 3.9 for some examples). If F is an exponential field with cyclic kernel which satisfies SC, then the rules of exponentiation constrain the behaviour of E so strongly that one can find a embedding of SK into F. This embedding is unique modulo sending $2\pi i$ to either τ or $-\tau$, so we will identify the image of the embedding with SK itself (thus identifying τ with $2\pi i$). The Schanuel condition then implies that $SK \lhd F$.

Schanuel's conjecture for $\mathbb{C}$ is equivalent to the statement that the complex exponential field $\mathbb{C}$ satisfies the Schanuel condition. It can easily be shown that Schanuel's conjecture is also equivalent to the assertion that $SK \lhd \mathbb{C}$.

3.3. **Exponential algebraic closure and CCP.** Given any exponential field F satisfying the Schanuel condition and any finite $X \subset F$ the function

$$\delta_F(X) := \text{trans.deg.}(X, \mathrm{E}(X)/\mathbb{Q}) - \text{lin.dim.}_{\mathbb{Q}}(X/0)$$

is always greater than or equal to 0. Now, it may happen that $\delta_F(X) > \delta_F(X \cup Y)$ but there can be no infinite descent, so we can define $\mathrm{etd}_F(X)$, the *exponential transcendence degree* of X in F, to be the minimum of $\delta_F(X_1)$ where $X_1 \supset X$. (Both X_1 and X are assumed to be finite subsets of F.)

For any finite $X \subset F$ we define the *exponential algebraic closure of X with respect to F*, denoted $\mathrm{ecl}_F(X)$, to be the set of all elements c such that $\mathrm{etd}_F(\{c\} \cup X) = \mathrm{etd}_F(X)$. For infinite X, we define $\mathrm{ecl}_F(X) = \bigcup\{\mathrm{ecl}_F(X_0) \mid X_0 \subseteq X, \text{ finite}\}$. We say that F satisfies the countable closure property (CCP) if the closure $\mathrm{ecl}_F(X)$ of every countable subset $X \subset F$ is countable. Notice that given any $X \subset F$, the exponential algebraic closure of X in F is an exponential field. The reader may care to look at [4] for an approach which does not rely on the Schanuel condition.

3.4. **Definition of Zilber fields.** Recall that in the introduction we defined Zilber fields as E-fields which are algebraically closed fields with standard kernel, surjective exponential map, which also satisfy the conclusion of Schanuel's conjecture, and which are *strongly exponentially-algebraically closed* – a notion which we have not defined yet. We now give the definition for the sake of completeness, although we do not use it directly in the paper.

Let F be any exponential field, let K be a subfield of F, and let $\alpha_1, \ldots, \alpha_n \in F$. Suppose (all other cases reduce to this) that the $\mathbb{Q}$-linear dimension of $\{\alpha_1, \ldots, \alpha_n\}$ is n. Let $V(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be the algebraic locus of $(\alpha_1, \ldots, \alpha_n, \mathrm{E}(\alpha_1), \ldots, \mathrm{E}(\alpha_n))$

over K. Then $(\alpha_1, \ldots, \alpha_n, \mathrm{E}(\alpha_1), \ldots, \mathrm{E}(\alpha_n))$ is a generic point of V over K. Moreover, V is a subvariety of $(\mathbb{G}_a)^n \times (\mathbb{G}_m)^n$.

Now, V has some special algebro-geometric properties. Firstly, the x coordinates of a generic point are $\mathbb{Q}$-linearly independent. Secondly, any "monomial" relation $\mathrm{E}(\alpha_1)^{m_1} \cdot \mathrm{E}(\alpha_2)^{m_2} \cdot \cdots \cdot \mathrm{E}(\alpha_n)^{m_n} = \beta$ (with $m_j \in \mathbb{Z}$ for all j) implies

$$\sum m_j \alpha_j = \delta$$

for some δ with $\mathrm{E}(\delta) = \beta$. ($\delta$ is defined only up to translation by elements of Ker.)

If there is in fact such a relation, we can reduce the study of $(\alpha_1, \ldots, \alpha_n)$ (and V) to a case of smaller n. Thus it makes sense to assume about $\overline{\alpha} := (\alpha_1, \ldots, \alpha_n)$ that there are no such relations.

Following Zilber, we call these assumptions on the $\bar{x}$ and $\bar{y}$ coordinates of V, *free from additive dependencies* and *free from multiplicative dependencies*, respectively. If V satisfies both conditions we just say it is *free*.

The Schanuel condition yields another constraint on (generic points of) V. Assuming that V is free, we easily deduce from SC that the dimension of V is at least n. But more is true. Let M be an $r \times n$ matrix over $\mathbb{Z}$, of rank r. Then $M\overline{\alpha}^T$ is a $\mathbb{Q}$-linearly independent r-tuple. Consider the values of E on the elements of the r-tuple. These are monomials (depending only on M) in the $\mathrm{E}(\alpha_j)$ (the y_j in effect). Then SC implies that the transcendence degree of

$$M\overline{\alpha}^T \cup \{\text{the corresponding E's}\}$$

is greater than or equal to r.

If V has this property of generic points then we say it is *rotund*. (Zilber used the terms *normal* and *ex-normal*.)

Thus in order to understand types in exponential fields satisfying SC, one is inevitably led to varieties which are *rotund and free*.

We are finally able to define strongly exponentially-algebraically closed.

Definition. An exponential field F is a *strongly exponentially-algebraically closed* if, given any rotund and free V and any finitely generated subfield K of F over which V is defined, there is a point in $V(F)$ of the form $(\overline{\alpha}, \overline{\mathrm{E}(\alpha)})$ which is generic in V over K.

3.5. **Extending automorphisms.** The deepest model theory in Zilber's work has to do with quasiminimal excellence. To understand this one has to go beyond [16], and the material is bound to be hard for those who are not specialists in pure model theory. The main results of our paper can be quickly obtained using quasiminimal excellence, but we also indicate how they can also be obtained without it.

The key structural property of Zilber fields can be summarized as follows:

Proposition. *Suppose F is a Zilber field with CCP, and F_0 is a finitely-generated partial E-subfield of F which contains SK, such that $F_0 \lhd F$. Then any automorphism of F_0 extends to an automorphism of F. In particular, the statement holds for any countable Zilber field F.*

Sketch proof. This follows from the quasiminimal excellence of the class of Zilber fields and Theorem 3.3 in [6]. Zilber uses a relational language whereas we use function symbols and the notion of partial exponential fields to give a simpler presentation. The notion of quasiminimal excellence depends critically on the language

chosen, but one can translate from one language to the other to see that his proof does indeed work to prove our statement. □

We could avoid the use of excellence altogether. In the case where F is countable and algebraic over $D(F) \cup \mathrm{E}(D(F))$, the proposition is a special case of part of [3, Theorem 7.2(2)]. This case is enough for our purposes.

3.6. **Countable subfields.**

Lemma. *Let F be any Zilber field, and let $X \subset F$ be a countable set. Then there is a countable elementary subfield $F' \prec F$ such that $X \subset F'$ and F' is also a Zilber field.*

Proof. The result follows from Theorem 2 and section 5 of [5]. We sketch a simpler proof.

Without the requirement that F' is a Zilber field, the result would follow immediately from the downward Löwenheim-Skolem theorem for a countable theory. In order to obtain a Zilber field we may need to add generic solutions to the free and rotund algebraic varieties. The idea is to construct a chain of structures, each an elementary substructure of F and each of which contains the previous field and has algebraically generic realizations for the rotund and free varieties defined with parameters over the previous field. This is a routine process. After constructing such chain of fields, one can define F' to be the union, which will have all the necessary properties. □

3.7. **Proof of Theorem 2.** We begin by proving the theorem assuming we are in the countable case and then use this to complete the general result. The countable case will be proved by an automorphism argument.

First suppose that F is a countable Zilber field (or more generally, a Zilber field with CCP). We define an automorphism σ_1 of SK by defining $\sigma_1(2\pi i) = -2\pi i$. Note that this defines a unique automorphism, which restricts to σ_0 on $\mathbb{Q}^{\mathrm{ab}}$. Since $SK \lhd F$, Proposition 3.5 allows us to extend σ_1 to an automorphism of F. Now if $\alpha \in \mathbb{Q}^{\mathrm{ab}} \smallsetminus \mathbb{Q}^{\mathrm{rab}}$ then $\sigma_0(\alpha) \neq \alpha$, so α is not pointwise definable in K.

Now let $\alpha \in \mathbb{Q}^{\mathrm{alg}} \smallsetminus \mathbb{Q}^{\mathrm{ab}}$. Let $F_0 = SK(\alpha)$, with $D(F_0) = D(SK)$. Then, since α is algebraic over SK but not in SK, there is an automorphism σ_2 of F_0 which fixes SK pointwise, but does not fix α. Since F_0 is an algebraic extension of SK and the domain of exponentiation does not extend, the property $SK \lhd F$ implies immediately that $F_0 \lhd F$. Thus σ_2 extends to an automorphism of F, and α is not pointwise definable in F.

If F is an uncountable Zilber field and $\alpha \notin \mathbb{Q}^{\mathrm{rab}}$, Lemma 3.6 above shows that there is a countable Zilber field F' containing α and elementarily embedded in F. We have shown that α is not definable in F' which implies that α is not definable in F. That completes the proof of Theorem 2. □

3.8. **Orbits and definable points.** When F is a Zilber field with CCP, we have shown that an algebraic number is in $\mathbb{Q}^{\mathrm{rab}}$ if and only if its orbit under automorphisms of F is a singleton.

3.9. **Other exponential fields.** The proof of Theorem 2 uses only that SK admits the automorphism σ_1, and that F is built on top of it in such a homogeneous way that the Proposition 3.5 holds. For any non-zero algebraic number τ, we can construct a partial exponential field CK_τ which is like SK, but with this τ

as the generator of a cyclic kernel in place of the usual transcendental generator. Then we can construct a strongly exponentially-algebraically closed exponential field $\mathbb{B}_\tau$, analogous to $\mathbb{B}$ but with CK_τ in place of SK. In this case there are two possibilities for what the definable algebraic numbers are. Let f be the minimal polynomial of τ over $\mathbb{Q}^{\mathrm{ab}}$, and let $\bar{f}$ be the polynomial obtained from it by applying the automorphism σ_0 of $\mathbb{Q}^{\mathrm{ab}}$ to its coefficients. If $\bar{f}(-\tau) = 0$ (for example, if $\tau = i$) then σ_0 extends to an involution on the partial exponential field CK_τ, and the definable algebraic numbers in $\mathbb{B}_\tau$ are those in the fixed field of that involution. Otherwise (for example, if $\tau = 1$), CK_τ has no non-trivial automorphisms, and the definable algebraic numbers are precisely the elements of CK_τ, that is, of $\mathbb{Q}^{\mathrm{ab}}(\tau)$.

Similarly, one can build exponential fields on SK (or on CK_τ) which are not strongly exponentially-algebraically closed, but still have the required homogeneity properties for the proof of Theorem 2 to go through, such as SK^{EA}, the free completion of SK to an algebraically closed exponential field, and SK^{ELA}, the free completion to an algebraically closed exponential field with logarithms. See [3] for details of all these constructions.

3.10. **Extending involutions.** Although the involution σ_1 on SK extends to some automorphism of $\mathbb{B}$, the extension is totally non-canonical, and the question of whether it can be extended to an involution on $\mathbb{B}$ is open and appears to be very difficult.

References

[1] H. Gonshor. *An introduction to the theory of surreal numbers*, volume 110 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1986.
[2] N. Jacobson. *Lectures in abstract algebra. III.* Springer-Verlag, New York, 1975. Theory of fields and Galois theory, Second corrected printing, Graduate Texts in Mathematics, No. 32.
[3] J. Kirby. Finitely presented exponential fields. *arXiv:0912.4019*, 2009.
[4] J. Kirby. Exponential algebraicity in exponential fields. *Bull. Lond. Math. Soc.*, 42(5):879–890, 2010.
[5] J. Kirby. A note on the axioms for Zilber's pseudo-exponential fields. *arXiv:1006.0894*, 2010.
[6] J. Kirby. On quasiminimal excellent classes. *J. Symbolic Logic*, 75(2):551–564, 2010.
[7] M. Laczkovich. The removal of π from some undecidable problems involving elementary functions. *Proc. Amer. Math. Soc.*, 131(7):2235–2240 (electronic), 2003.
[8] A. Macintyre. Surreal numbers. *Unpublished notes*.
[9] A. Macintyre. The complex exponential field is not model complete. *Unpublished notes*, 1991.
[10] A. Macintyre and A. J. Wilkie. On the decidability of the real exponential field. In *Kreiseliana*, pages 441–467. A K Peters, Wellesley, MA, 1996.
[11] D. Marker. A remark on Zilber's pseudoexponentiation. *J. Symbolic Logic*, 71(3):791–798, 2006.
[12] A. Tarski. *A decision method for elementary algebra and geometry.* University of California Press, Berkeley and Los Angeles, Calif., 1951. 2nd ed.
[13] L. van den Dries and P. Ehrlich. Fields of surreal numbers and exponentiation. *Fund. Math.*, 167(2):173–188, 2001.
[14] L. van den Dries, A. Macintyre, and D. Marker. Logarithmic-exponential power series. *J. London Math. Soc. (2)*, 56(3):417–434, 1997.
[15] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.
[16] B. Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic zero. *Ann. Pure Appl. Logic*, 132(1):67–95, 2005.